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RAN-2101001102030001

M. A. (Mathematics) (Part -II External) Examination March - 2026

Mathematics

Abstract Algebra (Paper 503)

Time: 3 Hours] **[Total Marks: 100**

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book (2) Attempt all questions. (3) Each question carries equal marks. (4) Follow usual notations and conventions. (5) Use of non-programmable scientific calculator is permitted.	Student's Signature

Q. 1. (a) Let $T = N_1 \times N_2 \times \cdots \times N_n$ and let a group G be the internal direct product of $N_1, N_2, ..., N_n$. Then prove that G and T are isomorphic.

(b) Find the possible number of 11–Sylow subgroups and 13–Sylow subgroups in a group G of order $11^2 \times 13^2$.

OR

Q. l. (a) Prove that the number of p–Sylow subgroups of a finite group G is of the form $1 + kp$, where p is prime.

(b) Show that any group of order 72 always have a non-trivial normal subgroup.

Q. 2. (a) State and prove Sylow's first theorem.

(b) State and prove the Eisenstein irreducibility Criterion of a polynomial with integer coefficients.

OR

Q. 2. (a) State and prove Gauss' lemma for the primitive polynomial.

(b) If R is aunique factorization domain, then show that $R\,[x_1, x_2, ..., x_n]$ is also a unique factorization domain.

Q. 3. (a) Prove that a group G is solvable if and only if $G^{(k)} = (e)$ for some $k \in \mathbb{N}$.

(b) If the real number α is constructible then show that α lies in some extension of the rationals of degree a power of 2. Also show that it is not possible to duplicate the cube by using only straight edge and compass.

OR

Q. 3. (a) Let L be a finite extension of K and K be a finite extension of F. Then show that L is a finite extension of F and $[L : K]\,[K : F] = [\mathrm{L} : F]$.

(b) Define normal extension. If F is a field of rational numbers and $K = F(\sqrt[3]{2})$), where $\sqrt[3]{2}$ is the real cube root of 2, then compute $G\,(K, F)$. Is K the normal extension of F? Justify.

Q. 4. (a) Let R be a commutative ring with unit element and M be an ideal of R. Prove that M is a maximal ideal of R if and only if R/M is a field.

(b) Let R be the ring of all the real-valued, continuous functions on the closed unit interval and let $M = \left\{ f(x) \in \mathrm{R} \,\middle|\, f\left(\frac{1}{2}\right) = 0 \right\}$. Then show that M is a maximal ideal of R.

OR

Q. 4. (a) If U is an idealof a ring R, then showthat R/U is a ring.

(b) Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Then show that R is a field.

Q. 5. (a) Let R be a ring with unity. Then prove that the R-module M is simple if and only if $M \simeq \frac{R}{I}$, where I is a maximal left ideal of R.

(b) Define: Cyclic R-module. If $1 \in R$ and M is an R-module generated by $\{x_1, x_2, ..., x_n)$, then show that $M = \{r_1x_1 + r_2x_2 + \cdots + r_nx_n \mid r_i \in R\}$.

OR

Q. 5. (a) Let M be finitely generated free module over a commutative ring R. Then prove that all basis of M have the same number of elements.

(b) Let A and B be R-submodules of R-modules M and N respectively.
Then prove that $\frac{M \times N}{A \times B} \simeq \frac{M}{A} \times \frac{N}{B}$.